RESEARCH ARTICLE

Airships: 100 Years of a Slide Rule, Thermodynamic Stability Equations Re-casted

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Abstract

The theoretical fundamentals of aerostatics governing airship flight are revisited and presented in a simple manner. In the analysis it is assumed that the airship is flying in a standard atmosphere, and that both the atmospheric air and the gas inside the envelope of the airship are governed by a polytropic thermodynamic process. The equations formulated in this paper are used to solve some of the problems that were formulated in 1923 by the designers of a slide rule. The slide rule was used by the north-american pilots of the "Blimps" in the 1920s. The models presented in this paper should be considered not only to calculate the aerostatic equilibrium of the new generation airships, but also to evaluate the aerostatic performance of the balloons, that will be used in the future, to transport experimental devices to the surface of the terrestrial planets.

Keywords: airship aerostatic, thermodynamic, blimp conditions, standard atmosphere, taut conditions, outer space balloons.

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1 Introduction

The airship technology was born at the end of the XIX century and reached the stage of "full" development during the first three decades of the XX century. The dramatic incident of the German airship "Hindenburg", occurred in Lakehurst, New Jersey, USA, on May 6th, 1937, was the end of the first era of the "giant flyers", see [1–6].

The recent development of computational systems, used by scientists and engineers to solve fluid dynamics equations for predicting atmospheric wind behavior, now enables finding optimal solutions to problems faced by those who enabled the early military and commercial airship flights.

The next generation of airships should be considered not only as competitors of the conventional air transportation systems, but due to their inherent characteristics (secure, low operation costs, stability and long range flight independence), they must be considered as aircrafts that can be efficiently used in applications such as [7–14]:

- Observance: Aerial photograph, police surveillance, coast guard surveillance, forest fire detection, vehicular trafic monitoring, atmospheric contaminants measurement, surveillance of oil ducts, mineral sources

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detection.

- Communication: Long range high-altitude aerial platforms for the transmission of radio, TV stations and internet.
- Transportation: goods, food, rescue of victims.
- Scientific research: Meteorology, geophysics, oceanography, science balloons soaring, and research of the animal wild life.
- Military applications: Anti-submarine warfare and search of maritime mines.

Very recently, the use of balloons (whose aerostatic behaviour is similar to that of the airships) has been considered in several spacial missions aimed to transport sensitive instruments to the surface of the terrestrial planets (planetary balloon missions). An understanding of the aerostatic of the balloons within the atmosphere of the terrestrial planets, is very important to guarantee the success of the missions, see for instance http://www.gaerospace.com/space-exploration/planetary-balloons/mars-balloons/ or https://vfm.jpl.nasa.gov/venusdesignreferencemission/vdrmballoons2/.

2 Aerostatics of the airships

To understand the aerodynamics of an airship, it is important to know the theoretical fundamentals that govern the aerostatics of these aircrafts.

The balance of the forces acting on an airship, in the vertical direction z (where the positive axis z is directed upward), can be expressed as

$$F_R = F_B - F_T, \tag{1}$$

where F_T is the total weight of the aircraft (crew, ballast, engines, fuel, envelope, etc.), F_B is the buoyancy force (due to the volume occupied by the gas within the airship, in the atmospheric air), and F_R is the vertical resultant force, or the lifting power of the ship (as it was named by [15] and [16]). The vertical force balance is present independently of the aerodynamic forces acting on the aircraft as a result of the flight (that is the trust and drag forces).

The aerostatic equilibrium of the airship depends on the vertical variation of the atmospheric air temperature dT_{air}/dz, and the balance between the vertical variation of the atmospheric pressure and the gravity force, that is

$$\frac{dp}{dz} = -\rho_{air} g = -\gamma_{air}, \tag{2}$$

where p [N/m^2], ρ_{air} [kg/m^3] and γ_{air} are the pressure, density and weight per unit volume of the atmospheric air respectively, and g [m/s^2] is the gravity acceleration.

During the first flights of the american Blimps (that is at the beginning of the 1920s), the pilots of the airships used slide rules to calculate the aerostatic equilibrium of the aircraft, see [16] and [15]. The thermodynamics fundamentals, on which the design of the slide rules was based, are presented in this paper, see [15] and [17].

The mathematical expressions presented in this paper (which are based on the thermodynamical models shown by [17]), are successfully validated by solving eight of the eleven problems that were proposed and solved by [15].

It may be concluded that the simple mathematical models revisited and presented in this paper, can be used (by undergraduate students in physics and engineering)) to calculate, not only the aerostatic behaviour of airships, but also the determine the aerostatic equilibrium of high altitude and outer space balloons.

In the solution of the problems proposed by [15], it is assumed in this paper, that the airship is flying in a standard atmosphere.

2.1 The Universal Standard Atmosphere

The Universal Standard Atmosphere is considered as an hypothetical vertical distribution of the pressure, temperature and density of the atmospheric air. In the Universal Standard Atmosphere model, it is assumed that the atmospheric air: (i) is an ideal gas, hence its thermodynamics behaviour is governed by the following state equation

$$\frac{p}{\rho_{air} g} = \frac{R_{air}}{g} T_{air}, \tag{3}$$

or

$$\frac{p}{\gamma_{air}} = p v_{air} = B_{air}\, T_{air}, \tag{4}$$

and (ii) it satisfies the hydrostatic equilibrium, see Eq. (2). In Eqs. (3) and (4), T_{air} [K] is the temperature of the atmospheric air, R_{air}=287.053 J/kg-K is the constant of the air, which is evaluated from $R_{air} = R/M_{air}$, R is the universal gas constant, that is 8314.51 J/(kgmole-K) and M_{air} is the average molar mass of the air, that is 28.965 kg/kgmole, $B_{air} = R_{air}/g$=29.271 m/K, where g=9.80665 m/s^2. Notice that $v_{air} = 1/\gamma_{air}$ [m^3/N].

In the Universal Standard Atmosphere model, it is assumed that the air is free of humidity, and that the gravity acceleration g does not change with respect to the vertical direction z. In the definition of the Universal Standard Atmosphere, several physical parameters are considered, however for the analysis of the buoyancy of the airships (and for the solution of the problems proposed by [15]), the values of the parameters that we use are the following

1. Constant of the air (without humidity), R_{air}=287.053 J/kg-K.
2. Gravity acceleration, g=9.80665 m/s^2.
3. Variation of the temperature with respect to the vertical direction z (environmental lapse rate) $-dT_{air}/dz$=0.0065 K/m.
4. The polytropic coefficient of the air (n) is calculated from

$$\begin{aligned} n &= \frac{1}{1 + B_{air}\frac{dT_{air}}{dz}} \\ &= \frac{1}{1 + [(29.271)(-0.0065)]} \\ &= 1.2349. \end{aligned} \tag{5}$$

2.2 The atmospheric air as a simple compressible substance

In the analysis, it is assumed that the atmospheric air is a simple compressible substance, that is the only permissible (reversible) works are expansion or compression. The thermodynamics state of a simple compressible substance, is fully determined from the specification of two independent thermodynamical properties [18].

In this paper, it is assumed that the thermodynamic behaviour of the atmospheric air is governed by a polytropic process (in which there exists a hydrostatic equilibrium and the temperature gradient dT_{air}/dz is constant) which is defined by the following relationship

$$p\left(\frac{1}{\rho_{air} g}\right)^n = p\left(\frac{1}{\gamma_{air}}\right)^n = p v_{air}^n = \text{const.} \tag{6}$$

Then, if the ideal gas equation (see Eq. (3)) is used together with the hydrostatic equation, see Eq. (2), it is obtained that the variation with respect to the vertical coordinate z of the pressure p , temperature T and specific weight γ of the air is given as, see [17]

$$p = p_1\left[1 + \frac{dT_{air}}{dz}\frac{z}{T_{air_1}}\right]^{\frac{n}{n-1}} = p_1\left[1 - az\right]^{\frac{n}{n-1}}, \tag{7}$$

$$T_{air} = T_{air_1} + \frac{dT_{air}}{dz}z = T_{air_1}(1 - az) \tag{8}$$

and

$$\gamma_{air} = \gamma_{air_1}\left[1 + \frac{dT_{air}}{dz}\frac{z}{T_{air_1}}\right]^{\frac{1}{n-1}} = \gamma_{air_1}\left[1 - az\right]^{\frac{1}{n-1}}, \tag{9}$$

where it is observed that the coefficient a is defined as

$$a = -\frac{1}{T_{air_1}}\frac{dT}{dz}, \tag{10}$$

p_1, T_{air_1} and γ_{air_1} are known values of the atmospheric air at the reference height z_1.

In the aerostatic analysis of an airship, the buoyancy force, in Eq. (1), is given as

$$F_B = (\gamma_{air} - \gamma_{gas})\,V, \tag{11}$$

where V is the volume of gas confined within the envelope of the airship. Notice that the subscript gas refers to the gas properties. In the analysis, it is assumed that the gas (either hydrogen or helium), behaves also as an ideal gas, then R_{gas} is defined as

$$R_{gas} = \frac{R}{M_{gas}} \equiv \text{J/kg-K}, \tag{12}$$

where for hydrogen M_{gas}=2.016 kg/kgmole, while for helium M_{gas}=4.003 kg/kgmole.

The specific weights of the air and the gas that appear in Eq. (11) are defined as $\gamma_{air} = \rho_{air} g$ and $\gamma_{gas} = \rho_{gas} g$, respectively.

Two aerostatic conditions of the airship are analyzed in this paper, in the first condition, it is assumed that airship is in the "taut" state, that is the airship is full of gas. In the second condition, it is assumed that the airship is in the flexible "limp" state, that is the airship is partially filled with gas. The "limp" word was taken by the British and the USA Navy, to give name to the flexible airships as "Blimps".

In the aerostatic analysis of the "taut" and "limp" conditions, it is assumed that the pressure of the gas is equal to the local pressure (at the height z) of the atmospheric air (hence, $p_{gas} = p_{air} = p$, see Eqs. (6) and (7)), and that the expansion of the gas is also governed by a polytropic process, hence the gas satifies the following equation

$$p\left(\frac{1}{\gamma_{gas}}\right)^k = \text{const}, \tag{13}$$

where k is the gas polytropic coefficient. From the relationship:

$$\frac{p}{p_1} = \left(\frac{\rho_{gas}}{\rho_{gas_1}}\right)^k, \tag{14}$$

it is obtained

$$\begin{aligned}\gamma_{gas} &= \gamma_{gas_1}\left[1+\frac{dT_{air}}{dz}\frac{z}{T_{air_1}}\right]^{\frac{n}{k(n-1)}} \\ &= \gamma_{gas_1}\left[1-az\right]^{\frac{n}{k(n-1)}}.\end{aligned} \tag{15}$$

2.3 Aerostatics of an airship in "taut" condition

In this condition, it is assumed that the aircraft is totally full of gas, and that the volume of the airship remains constant, independently of the ascending or descending maneuverings, however, the mass of the gas confined within the envelope may change (for instance when gas is released to the atmosphere). The hydrostatic stability of the airship is obtained when Eq. (1) is derived with respect to the vertical direction z (and considering that the volume of the airship remains constant), that is (see [17])

$$\frac{dF_R}{dz} = \frac{dF_B}{dz} - \frac{dF_T}{dz}. \tag{16}$$

In the analysis, it is assumed that the total weight of the airship F_T does not change with respect to the vertical direction, then its derivative with respect to z is equal to zero. Hence from Eq. (16), and deriving Eq. (11), we may write

$$\frac{dF_R}{dz} = \frac{dF_B}{dz} = V\frac{d}{dz}\left(\gamma_{air}-\gamma_{gas}\right), \tag{17}$$

which can be written as (using the hydrostatic equation, see Eq. (2))

$$\begin{aligned}\frac{dF_B}{dz} &= V\frac{dp}{dz}\frac{d}{dp}\left(\gamma_{air}-\gamma_{gas}\right) \\ &= -V\,\gamma_{air}\frac{d}{dp}\left(\gamma_{air}-\gamma_{gas}\right).\end{aligned} \tag{18}$$

Using Eqs. (6) and (13) it is possible to write

$$\ln\gamma_{air} = \frac{1}{n}\ln p \tag{19}$$

and

$$\ln\gamma_{gas} = \frac{1}{k}\ln p. \tag{20}$$

Taking the derivatives of Eqs. (19) and (20) we get

$$d\ln\gamma_{air} = \frac{1}{n}d\ln p \tag{21}$$

and

$$d\ln\gamma_{gas} = \frac{1}{k}d\ln p. \tag{22}$$

Using the derivative rules

$$\frac{d\ln\gamma}{d\gamma} = \frac{1}{\gamma} \quad \text{and} \quad \frac{d\ln p}{dp} = \frac{1}{p}, \tag{23}$$

in Eqs. (21) and (22), we obtain

$$\frac{d\gamma_{air}}{dp} = \frac{1}{n}\frac{\gamma_{air}}{p} \tag{24}$$

and

$$\frac{d\gamma_{gas}}{dp} = \frac{1}{k}\frac{\gamma_{gas}}{p}. \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (18), we get [17],

$$\frac{dF_B}{dz} = -V\frac{\gamma_{air}}{p}\left(\frac{\gamma_{air}}{n}-\frac{\gamma_{gas}}{k}\right). \tag{26}$$

or

$$\int_{z_o}^{\Delta z} dF_B = -\frac{V}{n}\int_{z_o}^{\Delta z}\frac{\gamma_{air}^2}{p}dz + \frac{V}{k}\int_{z_o}^{\Delta z}\frac{\gamma_{air}\gamma_{gas}}{p}dz \tag{27}$$

Using Eqs. (7), (9) and (15) in Eq. (27), we get

$$\begin{aligned}\int_{z_o}^{\Delta z} dF_B = &-\frac{V}{nz_o}\gamma_{air_o}\int_{z_o}^{\Delta z}\left[1-az\right]^{\frac{2-n}{n-1}}dz \\ &+\frac{V}{kz_o}\gamma_{gas_o}\int_{z_o}^{\Delta z}\left[1-az\right]^{\frac{k(1-n)+n}{k(n-1)}}dz\end{aligned} \tag{28}$$

where $z_o = p_o/\gamma_o$. Integrating Eq. (28), we have

$$\begin{aligned}\int_{z_o}^{\Delta z} dF_B = &\frac{V\gamma_{air_o}}{nz_o}\frac{(n-1)}{a}\left[(1-az)^{\frac{1}{n-1}}\right]_{z_o}^{\Delta z} \\ &-\frac{V\gamma_{gas_o}}{kz_o}\frac{k(n-1)}{an}\left[(1-az)^{\frac{n}{k(n-1)}}\right]_{z_o}^{\Delta z}\end{aligned} \tag{29}$$

If it is assumed that $z_o = 0$, we get

$$\begin{aligned}\int_{z_o}^{\Delta z} dF_B = &\frac{V\gamma_{air_o}}{nz_o}\frac{(n-1)}{a}\left[(1-a\Delta z)^{\frac{1}{n-1}}-1\right] \\ &-\frac{V\gamma_{gas_o}}{kz_o}\frac{k(n-1)}{an}\left[(1-a\Delta z)^{\frac{n}{k(n-1)}}-1\right].\end{aligned} \tag{30}$$

It is observed that

$$\frac{n-1}{nz_o a} = 1. \tag{31}$$

Using Eq. (31) in Eq. (30), we obtain

$$\begin{aligned} F_{B_{\Delta z}} - F_{B_o} = \Big[&\gamma_{air_o} (1 - a\Delta z)^{\frac{1}{(n-1)}} \\ &- \gamma_{gas_o} (1 - a\Delta z)^{\frac{n}{k(n-1)}}\Big] V \\ &- (\gamma_{air_o} - \gamma_{gas_o}) V, \end{aligned} \tag{32}$$

where $F_{B_{\Delta z}}$ and F_{B_o} are the buoyancy forces at a height Δz and at the ground $z = 0$, respectively. If in Eq. (32), it is assumed that $k = n$ we obtain

$$\begin{aligned} F_{B_{\Delta z}} - F_{B_o} = &(\gamma_{air_o} - \gamma_{gas_o}) (1 - a\Delta z)^{\frac{1}{(n-1)}} V \\ &- (\gamma_{air_o} - \gamma_{gas_o}) V. \end{aligned} \tag{33}$$

From Eq. (33), the ballast formula proposed by [17] is obtained. Let's assume that at the ground $z = 0$, the airship is in equilibrium, hence from Eq. (1) we have $F_{R_o} = 0$, then $F_{B_o} = F_{T_o}$. However at $z = 0$, the pilot of the airship drops the ballast ΔF_T. Then, at a certain height Δz the airship will be again in equilibrium, hence $F_{R_{\Delta z}} = 0$, then $F_{B_{\Delta z}} = F_{T_{\Delta z}}$. Where

$$F_{B_{\Delta z}} = F_{T_{\Delta z}} = F_{T_o} - \Delta F_T. \tag{34}$$

Using Eq. (34) in Eq. (33) we have

$$(F_{T_o} - \Delta F_T) - F_{T_o} = F_{B_o} (1 - a\Delta z)^{\frac{1}{(n-1)}} - F_{B_o} \tag{35}$$

where

$$F_{B_o} = (\gamma_{air_o} - \gamma_{gas_o}) V. \tag{36}$$

Then Eq. (35) is reduced to

$$-\Delta F_T = F_{B_o} \left[(1 - a\Delta z)^{\frac{1}{(n-1)}} - 1 \right] \tag{37}$$

Where the height Δz that the airship will rise after dropping the ballast can be obtained, that is

$$\Delta z = \frac{1}{a} \left[1 - \left\{ 1 - \frac{\Delta F_T}{F_{B_o}} \right\}^{n-1} \right], \tag{38}$$

remember that F_{B_o} is equal to the initial weight of the airship F_{T_o}.

An interesting situation appears when the airship is moored on the earth surface, or it is in takeoff or landing conditions. Under these conditions, ΔF_T, in Eq. (38), corresponds to the force that the crew must exerts to maintain the airship on land, that is when F_{R_o}=0. Then the force ΔF_T exerted by the crew gives an idea about the height that the airship will attain once the airship is released.

2.3.1 *Effect of temperature changes in the "taut" state*

From the buoyancy force equation at a given height z

$$F_B = (\gamma_{air} - \gamma_{gas}) V, \tag{39}$$

and from the state equation, see Eq. (4)

$$\gamma_{air} = \frac{p}{B_{air} T_{air}} \quad \text{and} \quad \gamma_{gas} = \frac{p}{B_{gas} T_{gas}}, \tag{40}$$

we may write

$$F_B = \frac{pV}{B_{air}} \left[\frac{1}{T_{air}} - \frac{\sigma}{T_{gas}} \right], \tag{41}$$

where

$$\sigma = \frac{B_{air}}{B_{gas}} = \left(\frac{p}{\gamma_{air} T_{air}} \right) \left(\frac{\gamma_{gas} T_{gas}}{p} \right) = \frac{\gamma_{gas} T_{gas}}{\gamma_{air} T_{air}} \tag{42}$$

In Eq. (42) it has been assumed that the pressure p of both the air and the gas is the same. Taking the derivative of Eq. (1) with respect to T_{gas}, keeping the volume V, the pressure p and the weight of the airship F_T as constants, and using Eqs. (41) and (42) we have

$$\frac{\partial F_R}{\partial T_{gas}} = \frac{\partial F_B}{\partial T_{gas}} = \frac{V\gamma_{gas}}{T_{gas}}. \tag{43}$$

Notice that the product $V\gamma_{gas}$ can be written as

$$\begin{aligned} V\gamma_{gas} &= V \left(\frac{\gamma_{air} - \gamma_{gas}}{\gamma_{air} - \gamma_{gas}} \right) \gamma_{gas} \\ &= \left[\frac{(\gamma_{air} - \gamma_{gas}) V}{\frac{\gamma_{air}}{\gamma_{gas}} - 1} \right] \\ &= \frac{F_B \frac{\gamma_{gas}}{\gamma_{air}}}{1 - \frac{\gamma_{gas}}{\gamma_{air}}} = \frac{F_B \sigma \frac{T_{air}}{T_{gas}}}{1 - \sigma \frac{T_{air}}{T_{gas}}} \end{aligned} \tag{44}$$

hence

$$\frac{\partial F_R}{\partial T_{gas}} = \frac{\partial F_B}{\partial T_{gas}} = \left(\frac{F_B}{T_{gas}} \right) \left(\frac{\sigma \frac{T_{air}}{T_{gas}}}{1 - \sigma \frac{T_{air}}{T_{gas}}} \right). \tag{45}$$

Similarly, taking the derivative of Eq. (1) with respect to T_{air}, and keeping the volume V, the pressure p and the weight of the airship F_T as constants, and using Eqs. (41) and (42) we get

$$\frac{\partial F_R}{\partial T_{air}} = \frac{\partial F_B}{\partial T_{air}} = -\left(\frac{F_B}{T_{air}} \right) \left(\frac{1}{1 - \sigma \frac{T_{air}}{T_{gas}}} \right). \tag{46}$$

The change of the buoyancy force is calculated as

$$dF_B = \frac{\partial F_B}{\partial T_{gas}} dT_{gas} + \frac{\partial F_B}{\partial T_{air}} dT_{air}. \tag{47}$$

For finite small temperature differences, Eq. (47) can be written as, see [17],

$$\Delta F_B = \frac{\partial F_B}{\partial T_{gas}} \Delta T_{gas} + \frac{\partial F_B}{\partial T_{air}} \Delta T_{air}. \tag{48}$$

Using Eqs. (45) and (46) in Eq. (48) it is obtained

$$\frac{\Delta F_B}{F_B} = \left(\frac{1}{1-\sigma \frac{T_{air}}{T_{gas}}} \right) \left[\sigma \left(\frac{\Delta T_{gas}}{T_{gas}} \right) \left(\frac{T_{air}}{T_{gas}} \right) - \frac{\Delta T_{air}}{T_{air}} \right]. \tag{49}$$

If in Eq. (49), it is considered that $T_{gas} = T_{air} = T$, we obtain

$$\frac{\Delta F_B}{F_B} = \frac{\sigma \Delta T_{gas} - \Delta T_{air}}{T(1-\sigma)} \tag{50}$$

Notice that either Eq. (49) or Eq. (50) can be used in the ballast equation, Eq. (38) (that is, when $k = n$) to obtain the height Δz that the airship will change with respect to the height z at which T_{gas}, T_{air}, ΔT_{gas} and ΔT_{air} are evaluated. Hence, the ballast equation Eq. (38), when $T_{gas} = T_{air} = T$ (and $\Delta F_B/F_B = \Delta F_T/F_{B_o}$), is written as

$$\Delta z = \frac{1}{a} \left[1 - \left\{ 1 - \left(\frac{\sigma \Delta T_{gas} - \Delta T_{air}}{T(1-\sigma)} \right) \right\}^{n-1} \right], \tag{51}$$

2.4 Aerostatics of an airship in flexible "limp" condition

In this operational condition, the gas within the envelope of the airship is free to expand or to compress (following a polytropic thermodynamical process) without the need to release gas. Then, the volume occupied by the gas can change, however the amount of gas inside the airship remains constant. If the weight of the gas inside the airship is calculated as

$$W_{gas} = \gamma_{gas} V, \tag{52}$$

where V is the volume occupied by the gas.

Using Eq. (52) in Eq. (1) we write

$$\begin{aligned} F_R &= (\gamma_{air} - \gamma_{gas}) V - F_T \\ &= (\gamma_{air} - \gamma_{gas}) \left(\frac{W_{gas}}{\gamma_{gas}} \right) - F_T \\ &= W_{gas} \left[\frac{\gamma_{air}}{\gamma_{gas}} - 1 \right] - F_T \end{aligned} \tag{53}$$

The hydrostatic stability of an airship in a "limp" state is obtained by taking the derivative of Eq. (53) with respect to the vertical coordinate z, and keeping W_{gas} and the total weight of the airship F_T as constant values, that is

$$\begin{aligned} \frac{dF_R}{dz} &= \frac{dF_B}{dz} = W_{gas} \frac{d}{dz} \left(\frac{\gamma_{air}}{\gamma_{gas}} - 1 \right) \\ &= W_{gas} \frac{dp}{dz} \frac{d}{dp} \left[\frac{\gamma_{air}}{\gamma_{gas}} - 1 \right] \end{aligned} \tag{54}$$

Following the same procedure from which it was obtained Eq. (26), it is possible to write the following expression [17],

$$\frac{dF_B}{dz} = -\frac{W_{gas} \gamma_{air}^2}{\gamma_{gas} p} \left(\frac{1}{n} - \frac{1}{k} \right). \tag{55}$$

Using Eqs. (7), (9) and (15) in Eq. (55) we get

$$\begin{aligned} \int_{z_o}^{\Delta z} dF_B = &-\frac{W_{gas} \gamma_{air_o}}{\gamma_{gas_o} z_o} \left(\frac{1}{n} - \frac{1}{k} \right) \\ &\times \int_{z_o}^{\Delta z} (1 - az)^{\frac{k(2-n)-n}{k(n-1)}} dz. \end{aligned} \tag{56}$$

Integrating Eq. (56), we get

$$\begin{aligned} \int_{z_o}^{\Delta z} dF_B = &\frac{W_{gas} \gamma_{air_o}}{\gamma_{gas_o} z_o} \left(\frac{1}{n} - \frac{1}{k} \right) \\ &\times \left(\frac{k(n-1)}{a(k-n)} \right) \\ &\times \left[(1 - az)^{\frac{k-n}{k(n-1)}} \right]_{z_o}^{\Delta z} \end{aligned} \tag{57}$$

If it is assumed that z_o=0, we get

$$\begin{aligned} \int_{z_o}^{\Delta z} dF_B = &\frac{W_{gas} \gamma_{air_o}}{\gamma_{gas_o} z_o} \left(\frac{1}{n} - \frac{1}{k} \right) \\ &\times \left(\frac{k(n-1)}{a(k-n)} \right) \\ &\times \left[(1 - a\Delta z)^{\frac{k-n}{k(n-1)}} - 1 \right] \end{aligned} \tag{58}$$

From Eq. (31) it is observed that $a = (n-1)/(nz_o)$, then we can write the following expression

$$\frac{k(n-1)}{a(k-n)} = \frac{knz_o}{(k-n)}, \tag{59}$$

Using Eq. (59) in Eq. (58), we get

$$\begin{aligned} F_{B_{\Delta z}} - F_{B_o} = &\frac{knW_{gas} \gamma_{air_o}}{(k-n) \gamma_{gas_o}} \left(\frac{1}{n} - \frac{1}{k} \right) \\ &\times \left[(1 - a\Delta z)^{\frac{k-n}{k(n-1)}} - 1 \right] \\ = &\frac{W_{gas} \gamma_{air_o}}{\gamma_{gas_o}} \left[(1 - a\Delta z)^{\frac{k-n}{k(n-1)}} - 1 \right], \end{aligned} \tag{60}$$

where $F_{B_{\Delta z}}$ and F_{B_o} are the buoyancy forces at the height Δz and at the ground $z = 0$. Equation (60) can also be written as

$$F_{B_{\Delta z}} - F_{B_o} = W_{gas}\left[\frac{\gamma_{air_o}\left(1 - a\Delta z\right)^{\frac{1}{(n-1)}}}{\gamma_{gas_o}\left(1 - a\Delta z\right)^{\frac{n}{k(n-1)}}}\right] - \frac{W_{gas}\gamma_{air_o}}{\gamma_{gas_o}} \tag{61}$$

Using Eqs. (9) and (15) in Eq. (61) we have

$$F_{B_{\Delta z}} - F_{B_o} = W_{gas}\left(\frac{\gamma_{air}}{\gamma_{gas}}\right) - W_{gas}\left(\frac{\gamma_{air_o}}{\gamma_{gas_o}}\right). \tag{62}$$

Resting and summing W_{gas} in the first and in the second terms on the right hand side of the Eq. (62), it is obtained

$$F_{B_{\Delta z}} - F_{B_o} = W_{gas}\left[\left(\frac{\gamma_{air}}{\gamma_{gas}}\right) - 1\right] - W_{gas}\left[\left(\frac{\gamma_{air_o}}{\gamma_{gas_o}}\right) - 1\right], \tag{63}$$

which are the expressions for the buoyancy forces at the height Δz and $z = 0$, see Eq. (53).

A ballast formula can also be obtained from Eq. (60). Again, let's assume that at the ground, $z = 0$, the airship is in equilibrium, hence we have $F_{R_o} = 0$, then $F_{B_o} = F_{T_o}$. At $z = 0$ the crew drops the ballast ΔF_T, in such away that at the height Δz the airship will be again in equilibrium, then $F_{R_{\Delta z}} = 0$, hence $F_{B_{\Delta z}} = F_{T_{\Delta z}}$, but $F_{B_{\Delta z}} = F_{T_{\Delta z}} = F_{T_o} - \Delta F_T$. Using these last relationships in Eq. (60) we have

$$\left(F_{T_o} - \Delta F_T\right) - F_{T_o} = W_{gas}\left(\frac{\gamma_{air_o}}{\gamma_{gas_o}}\right) \times \left[\left(1 - a\Delta z\right)^{\frac{k-n}{k(n-1)}} - 1\right]. \tag{64}$$

from which it is obtained

$$\Delta z = \frac{1}{a}\left[1 - \left\{1 - \frac{\Delta F_T}{W_{gas}\left(\frac{\gamma_{air_o}}{\gamma_{gas_o}}\right)}\right\}^{\frac{k(n-1)}{k-n}}\right]. \tag{65}$$

2.5 Effect of temperature changes in the "limp" state

Using Eq. (42) in Eq. (53) the resultant force F_R in the limp state is written as

$$F_R = W_{gas}\left[\frac{1}{\sigma}\frac{T_{gas}}{T_{air}} - 1\right] - F_T. \tag{66}$$

Taking the derivative of Eq. (66) with respect to T_{air}, and keeping the weight of the gas W_{gas}, the pressure p and the weight of the airship F_T as constants we have

$$\frac{\partial F_R}{\partial T_{air}} = \frac{\partial F_B}{\partial T_{air}} = -\frac{W_{gas}T_{gas}}{\sigma T_{air}^2}. \tag{67}$$

Using Eqs. (42) and (52) in Eq. (67) we get

$$\frac{\partial F_B}{\partial T_{air}} = -\frac{V\gamma_{air}}{T_{air}}, \tag{68}$$

where $V\gamma_{air}$ is written as

$$V\gamma_{air} = V\frac{(\gamma_{air} - \gamma_{gas})}{(\gamma_{air} - \gamma_{gas})}\gamma_{air} = \frac{F_B}{1 - \frac{\gamma_{gas}}{\gamma_{air}}} = \frac{F_B}{1 - \sigma\frac{T_{air}}{T_{gas}}}. \tag{69}$$

Using Eq. (69) in Eq. (68), it is obtained

$$\frac{\partial F_B}{\partial T_{air}} = -\frac{F_B}{T_{air}}\left[\frac{1}{1 - \sigma\frac{T_{air}}{T_{gas}}}\right]. \tag{70}$$

Similarly, taking the derivative of Eq. (66) with respect to T_{gas}, and keeping the weight of the gas W_{gas}, the pressure p and the weight of the airship F_T as constants we have

$$\frac{\partial F_B}{\partial T_{gas}} = \frac{W_{gas}}{\sigma T_{air}} = \frac{W_{gas}}{\left(\frac{\gamma_{gas}}{\gamma_{air}}\frac{T_{gas}}{T_{air}}\right)T_{air}} = \frac{V\gamma_{gas}\gamma_{air}}{\gamma_{gas}T_{gas}} = \frac{V\gamma_{air}}{T_{gas}}. \tag{71}$$

Using Eq. (69) in Eq. (71), we get

$$\frac{\partial F_B}{\partial T_{gas}} = \frac{F_B}{T_{gas}}\left[\frac{1}{1 - \sigma\frac{T_{air}}{T_{gas}}}\right]. \tag{72}$$

For finite small temperature differences (see Eq. (47)), and using Eqs. (70) and (72) in Eq. (48), we get

$$\frac{\Delta F_B}{F_B} = \left(\frac{1}{1 - \sigma\frac{T_{air}}{T_{gas}}}\right)\left[\frac{\Delta T_{gas}}{T_{gas}} - \frac{\Delta T_{air}}{T_{air}}\right]. \tag{73}$$

If in Eq. (73) it is considered that $T_{gas} = T_{air} = T$, we obtain

$$\frac{\Delta F_B}{F_B} = \frac{\Delta T_{gas} - \Delta T_{air}}{T\left(1 - \sigma\right)}. \tag{74}$$

Notice that either Eq. (73) or Eq. (74) can be used in the ballast equation, Eq. (65), where the term $W_{gas}\gamma_{air}/\gamma_{gas}$, can be written as (see Eq. (69))

$$W_{gas}\left(\frac{\gamma_{air}}{\gamma_{gas}}\right) = V\gamma_{gas}\left(\frac{\gamma_{air}}{\gamma_{gas}}\right) = V\gamma_{air} = \frac{F_B}{1 - \frac{\gamma_{gas}}{\gamma_{air}}} \tag{75}$$

Using Eq. (75) in the ballast formula Eq. (65) (when $\Delta F_B/F_B = \Delta F_T/F_B$), we have

$$\Delta z = \frac{1}{a}\left[1 - \left\{1 - \frac{\Delta F_B}{F_B}\left(1 - \frac{\gamma_{gas}}{\gamma_{air}}\right)\right\}^{\frac{k(n-1)}{k-n}}\right]. \quad (76)$$

When $T_{gas} = T_{air} = T$, the height Δz that the airship will change with respect respect to the height z at which T_{gas}, T_{air}, ΔT_{gas} and ΔT_{air} are evaluated, is obtained when Eq. (74) is used in Eq. (76), hence we have

$$\Delta z = \frac{1}{a}\left[1 - \left\{1 - \frac{(\Delta T_{gas} - \Delta T_{air})}{T(1-\sigma)}\left(1 - \frac{\gamma_{gas}}{\gamma_{air}}\right)\right\}^{\frac{k(n-1)}{k-n}}\right] \quad (77)$$

3 Results and Discussion

In this section we present the solution of the Weaver and Pickering (1923) proposed set of hydrostatic problems that were solved using the "slide rule", by the pilots of the US "Blimps", as a part of their training.

The use of slide rule was required by the USA Navy, due to the success of a previous slide rule named the Scott-Teed rule, that was used by the British airships pilots [15].

The solution of these real life situations/problems has been explained below as case studies. It is assumed that the gas within the envelope of the airship is hydrogen and that the atmosphere is a standard one. In the problems, it is assumed that the airship has a total volume of 6880.9 m3 (243,000 cubic feet).

Case study 1

To calculate the total lifting power of an airship of V=6880.99 m^3 (243,000 cubic feet) capacity at an altitude of 1524 m (5,000 feet), if the barometer reading at the ground is 101305.5 Pa (30 inches) and the air temperature 288.7 K (60 ^{o}F) and the hydrogen is 95 per cent pure.

It is assumed that the airship is in the "taut" state. The polytropic coefficients of the air and gas are n=1.2349 and k =1.4, respectively. The value of the coefficient a is evaluated from Eq. (10), then

$$a = -\frac{1}{T_o}\frac{dT}{dz} = -\frac{1}{288.7}(-0.0065) = 2.251\times10^{-5}\ 1/\text{m}. \quad (78)$$

On the surface of the earth (z_o=0 m) where p_o=101305.5 Pa and T_o=288.7 K, the specific weights of the air and the gas are calculated from the state equation, see Eq. (3), hence

$$\gamma_{air_o} = \frac{p_o g}{R_{air} T_o} = \frac{(101305.5)(9.80665)}{(287.04)(288.7)} = 11.98\ \text{N/m}^3, \quad (79)$$

and

$$\gamma_{gas_o} = \frac{p_o g}{R_{gas} T_o} = \frac{p_o g}{\frac{R}{M_{gas}} T_o} = \frac{(101305.5)(9.80665)}{\left(\frac{8314.51}{2.016}\right)(288.7)} = 0.834\ \text{N/m}^3. \quad (80)$$

The buoyancy force $F_{B_{\Delta z}}$ at the height $\Delta z = 1524$ m is calculated from Eq. (32), hence

$$F_{B_{\Delta z}} = \left[\gamma_{air_o}(1 - a\Delta z)^{\frac{1}{(n-1)}} - \gamma_{gas_o}(1 - a\Delta z)^{\frac{n}{k(n-1)}}\right] V = 66061.458\ \text{N}. \quad (81)$$

Literature [15] suggest to modify the buoyancy force by 0.95 (percent of the hydrogen purity) hence $F_{B_{\Delta z}}$ = (0.95)(66061.458)=62758.386 N. Literature [15] reported a value of $F_{B_{\Delta z}}$ = 62453 N (14,040 pounds). Then we have a percentage error equal to 0.48 %.

Case study 2

We calculate the initial gas mass in an airship in order to reach an altitude of 2438.4 m (8,000 feet) without losing gas.

As the airship is assumed to be in the "limp" state, the amount of gas (that is the weight of the gas W_{gas}) within the airship remains constant. From Eq. (52), the weight of the gas at the surface z=0, and at the height Δz= 2438.4 m, is given by

$$W_{gas} = \gamma_{gas_o} V_o \quad \text{and} \quad W_{gas} = \gamma_{gas_{\Delta z}} V_{gas_{\Delta z}}, \quad (82)$$

respectively. Equating both Eqs. (82) we get

$$\gamma_{gas_o} V_o = \gamma_{gas_{\Delta z}} V_{gas_{\Delta z}}, \quad (83)$$

from which the ratio $V_o/V_{gas_{\Delta z}}$ is obtained, that is

$$\frac{V_o}{V_{gas_{\Delta z}}} = \frac{\gamma_{gas_{\Delta z}}}{\gamma_{gas_o}}. \quad (84)$$

Taken the values of the parameters at the start to be the same as the values at the ground (z=0) of the problem 1, we obtain γ_{gas_o}=0.834 N/m^3, and $\gamma_{gas_{\Delta z}}$ is calculated from Eq. (15)

$$\gamma_{gas_{\Delta z}} = \gamma_{gas_o}[1 - a\Delta z]^{\frac{n}{k(n-1)}} = 0.6749\ \text{N/m}^3. \quad (85)$$

Hence

$$\frac{V_o}{V_{gas_{\Delta z}}} = \frac{\gamma_{gas_{\Delta z}}}{\gamma_{gas_o}} = 0.809. \tag{86}$$

The initial volume of the airship is 80.9% of the volume that it will have at an altitude of 2438.4 m (without lost of gas). Literature [15] report 78 %. This means that we have a percentage error equal to 3.71 %.

Case study 3

An airship is in equilibrium at a height of 609.6 m (2,000 feet). The pilot estimates that the ship is 90 per cent full and that the total weight of the ship and its load is 48930.4 N (11,000 pounds). We calculate how much ballast must be dropped to rise to a height of 1828.8 m (6,000 feet).

The airship is in the "limp" state. Then, the weight of the gas W_{gas} in the airship remains constant. From the balance force equation, see Eq. (53), at the height z_1= 609.6 m, the airship is in equilibrium F_R=0, then $F_B = F_T$, hence we have

$$(\gamma_{air_1} - \gamma_{gas_1}) V_1 = F_{T_1} \tag{87}$$

The airship is flying in a standard atmosphere. The parameters at the surface have the following values T_o=288.15 K, p_o=101325 Pa, g=9.80665 m/s^2, a=2.255 $\times 10^{-5}$ 1/m. Hence from the state equation the specific weights of the gas and the air are γ_{air_o}=12.012 N/m^3 and γ_{gas_o}=0.836 N/m^3, respectively.

where

$$\begin{aligned}\gamma_{air_1} &= \gamma_{air_o}\left[1 - az_1\right]^{\frac{1}{(n-1)}}\\ &= 12.012\left[1 - (2.255\times10^{-5})(609.6)\right]^{\frac{1}{(1.2349-1)}}\\ &= 11.32 \text{ N/m}^3.\end{aligned} \tag{88}$$

$$\begin{aligned}\gamma_{gas_1} &= \gamma_{gas_o}\left[1 - az_1\right]^{\frac{n}{k(n-1)}}\\ &= 0.836\left[1 - (2.255\times10^{-5})(609.6)\right]^{\frac{1.2349}{1.4(1.2349-1)}}\\ &= 0.7937 \text{ N/m}^3.\end{aligned} \tag{89}$$

Then V_1 can be obtained as

$$\begin{aligned}V_1 &= \frac{F_{T_1}}{(\gamma_{air_1} - \gamma_{gas_1})} = \frac{48930.4}{(11.32 - 0.7937)}\\ &= 4720.733 \text{ m}^3.\end{aligned} \tag{90}$$

From Eq. (63), it is possible to write

$$\begin{aligned}F_{B_2} - F_{B_1} = W_{gas}&\left[\left(\frac{\gamma_{air_2}}{\gamma_{gas_2}}\right) - 1\right]\\ - W_{gas}&\left[\left(\frac{\gamma_{air_1}}{\gamma_{gas_1}}\right) - 1\right]\end{aligned} \tag{91}$$

or

$$\begin{aligned}(F_{B_1} - \Delta F_T) - F_{B_1} &= W_{gas}\left[\left(\frac{\gamma_{air_2}}{\gamma_{gas_2}}\right) - \left(\frac{\gamma_{air_1}}{\gamma_{gas_1}}\right)\right]\\ &= \gamma_{gas_1} V_1\left[\left(\frac{\gamma_{air_2}}{\gamma_{gas_2}}\right) - \left(\frac{\gamma_{air_1}}{\gamma_{gas_1}}\right)\right],\end{aligned} \tag{92}$$

or

$$\Delta F_T = -V_1\left[\gamma_{gas_1}\left(\frac{\gamma_{air_2}}{\gamma_{gas_2}}\right) - \gamma_{air_1}\right], \tag{93}$$

where

$$\gamma_{air_2} = \gamma_{air_o}\left[1 - az_2\right]^{\frac{1}{(n-1)}} = 10.04 \text{ N/m}^3. \tag{94}$$

$$\gamma_{gas_2} = \gamma_{gas_o}\left[1 - az_2\right]^{\frac{n}{k(n-1)}} = 0.7137 \text{ N/m}^3. \tag{95}$$

Substituting the values of V_1, γ_{air_1}, γ_{gas_1}, γ_{air_2} and γ_{gas_2} in Eq. (93), it is obtained ΔF_T=901.8 N. Literature [15] report 934.12 N (210 pounds). Hence we have a percentage error equal to 3.45 %.

Case study 4

To calculate then the lifting power of a balloon (airship) with total load (in equilibrium) being 60050.97 N (13,500 pounds). It is assumed that the temperature of the air is 272.03 K (30^{o}F) and the temperature of the gas is 280.37 K (45^{o}F). It is also assumed that no gas is lost until after sunset, when the temperature of gas and air will become equal.

It is assumed that the airship is in "limp" state. From Eq. (73)

$$\Delta F_B = F_B\left(\frac{1}{1 - \sigma\frac{T_{air}}{T_{gas}}}\right)\left[\frac{\Delta T_{gas}}{T_{gas}} - \frac{\Delta T_{air}}{T_{air}}\right] \tag{96}$$

where T_{air}=272.03 K, T_{gas}=280.37 K, ΔT_{gas}=272.03-280.37=-8.33 K, ΔT_{air}=0 K, F_B=60050.97 N and σ=B_{air}/B_{gas}=29.271/420.557=0.0696. Then, ΔF_B =-1913.38 N. Hence the buoyancy force is

F_B=60050.97+(-1913.38)=58137.59 N. Literature [15] report 56803.76 N (12,770 pounds). Hence we have a percentage error equal to 2.34 %.

Case study 5

When an airship of 6880.99 m^3 (243,000 cubic feet) capacity reaches the summit of its flight, the barometer is observed to read 74301.33 Pa (22 inches), the temperature of the gas is 272.04 K (30^{o}F), and its purity 98 per cent. For the above situation lifting power of the ship is calculated when the air temperature is 283.15 K (50^{o}F) and the gas temperature is 291.48 K (65^{o}F).

Assuming that the airship is in the "limp" state, the height where the airship reaches the summit of its flight, the temperature of the gas is equal to the temperature of the air, that is T_{gas}=T_{air}=272.03 K. From the state equation, the specific weights of the gas and the air, at this height are calculated as

$$\gamma_{gas} = \frac{p}{B_{gas}T} = \frac{74301.33}{(420.55)(272.04)} = 0.649 \quad \text{N/m}^3, \tag{97}$$

$$\gamma_{air} = \frac{p}{B_{air}T} = \frac{74301.33}{(29.271)(272.04)} = 9.33 \quad \text{N/m}^3. \tag{98}$$

The weight of the gas at this height is calculated as

$$W_{gas} = V\gamma_{gas} = (6880.99)(0.649) = 4468.78 \text{ N}. \tag{99}$$

At the summit the buoyancy force is calculated as

$$\begin{aligned} F_B &= W_{gas}\left[\frac{\gamma_{air}}{\gamma_{gas}} - 1\right] \\ &= 4468.78\left[\frac{9.33}{0.649} - 1\right] \\ &= 59735.40 \text{ N} \end{aligned} \tag{100}$$

From Eq. (74)

$$\Delta F_B = F_B\left(\frac{\Delta T_{gas} - \Delta T_{air}}{T(1-\sigma)}\right), \tag{101}$$

where ΔT_{gas}=291.48-272.04=19.44 K, ΔT_{air}=283.15-272.04=11.11 K and σ=0.0696. Then from Eq. (74)

$$\Delta F_B = 59730.79\left(\frac{19.44 - 11.11}{272.04(1-0.0696)}\right) = 1966.67 \text{ N}. \tag{102}$$

Hence the buoyancy force is F_B=59735.40+1966.67=61702.075 N. However the purity of the gas is 98%, the we have F_B=0.98(61702.075)=60468.035 N. Literature [15] report 60629.23 N (13630 pounds). Hence we have a percentage error equal to 0.26 %.

Case study 6

To calculate the height of an airship of 6880.99 m^3 (243,000 cubic feet) capacity rise with a load of 40033.99 N (9,000 pounds) if it filled with 98 per cent hydrogen, the barometer reads 84433.33 Pa (25 inches), and the air temperature is 299.81 K (80^{o}F).

It is assumed the airship is in "limp" state. From Eq. (63),

$$F_{B_{\Delta z}} = W_{gas}\left[\left(\frac{\gamma_{air_{\Delta z}}}{\gamma_{gas_{\Delta z}}}\right) - 1\right]. \tag{103}$$

At the height z at which the measurements were taken and the volume is 98 % of the total volume V_T=6880.99 m^3, that is V_z=(0.98)V_T=(0.98)(6880.99)=6743.37 m^3, the gas specific weight is

$$\gamma_{gas_z} = \frac{p_z}{B_{gas}T_z} = \frac{84433.33}{(420.557)(299.80)} = 0.6696 \text{ N/m}^3, \tag{104}$$

while the air specific weight is

$$\gamma_{air_z} = \frac{p_z}{B_{air}T_z} = \frac{84433.33}{(29.271)(299.80)} = 9.62 \text{ N/m}^3. \tag{105}$$

The weight of the gas W_{gas}, which remains constant, is evaluated at the height z as

$$W_{gas} = \gamma_{gas_z}V_z = (0.6696)(6743.37) = 4515.64 \text{ N}. \tag{106}$$

The specific weight of the gas at the new vertical position Δz is calculated as

$$\gamma_{gas_{\Delta z}} = \frac{W_{gas}}{V_T} = \frac{4515.64}{6880.99} = 0.656 \text{ N/m}^3 \tag{107}$$

At the height Δz the buoyancy force of the airship $F_{B_{\Delta z}}$ is in equilibrium with the load 40033.99 N, then Eq. (103) can be written as

$$40033.99 = 4515.64\left[\left(\frac{\gamma_{air_{\Delta z}}}{0.656}\right) - 1\right], \tag{108}$$

from which, $\gamma_{air_{\Delta z}}$=6.474 N/m^3. Using this value of $\gamma_{air_{\Delta z}}$ in Eq. (9), we get

$$\gamma_{air_{\Delta z}} = \gamma_{air_z}\left[1 - a\Delta z\right]^{\frac{1}{n-1}}, \tag{109}$$

The height Δz is given as

$$\begin{aligned} \Delta z &= \frac{1}{a}\left[1 - \left(\frac{\gamma_{air_{\Delta z}}}{\gamma_{air_z}}\right)^{n-1}\right] \\ &= \frac{1}{2.251\times 10^5}\left[1 - \left(\frac{6.474}{9.62}\right)^{1.2349-1}\right] \\ &= 3946.96 \text{ m}. \end{aligned} \tag{110}$$

Literature [15] report 4023.36 N (13200 feet). Hence we have a percentage error equal to 1.89 %.

Case study 7

The total weight of the ship of 6880.9 m^3 (243,000 cubic feet) capacity and its load is 66723.3 N (15,000 pounds). It is just in equilibrium at a barometric pressure of 104697.33 Pa (31 inches) and an air temperature of 283.15 K (50^{o}F). The purity of the hydrogen is 94 per cent. We calculate the increase in liting power of the ship if pure hydrogen is used to fill the bag.

The airship does not change its vertical position and it is assumed to be in "limp" state. It is further assumed that the gas and the air have the same temperature, that is T_{gas_1}=T_{air_1}=T=283.15 K. Hence the specific weights are calculated as

$$\gamma_{gas} = \frac{p}{B_{gas}T} = \frac{104697.33}{(420.557)(283.15)} = 0.8792 \text{ N/m}^3, \tag{111}$$

and

$$\gamma_{air} = \frac{p}{B_{air}T} = \frac{104697.33}{(29.271)(283.15)} = 12.631 \text{ N/m}^3. \tag{112}$$

In equilibrium the buoyancy force is equal to the load (F_T=66723.3 N) then we have

$$F_B = V\left(\gamma_{air} - \gamma_{gas}\right) = F_T = 66723.3 \text{ N}, \tag{113}$$

Then, the volume occupied by the gas when it is 94% pure is obtained by

$$\begin{aligned} V_{94\%} &= \frac{F_B}{0.94\left(\gamma_{air} - \gamma_{gas}\right)} \\ &= \frac{66723.3}{0.94\left(2.631 - 0.8792\right)} \\ &= 6039.675 \text{ m}^3. \end{aligned} \tag{114}$$

If the airship is going to be filled now with pure hydrogen until the bag is full, the increment in volume must be $\Delta V_{100\%}$=V_T-$V_{94\%}$=6880.9-6039.675=841.314 m^3. Then the increment in buoyancy should be

$$\begin{aligned} \Delta F_B &= \Delta V_{100\%}\left(\gamma_{air} - \gamma_{gas}\right) \\ &= 841.314\left(12.631 - 0.879\right) \\ &= 9887.685 \text{ N}. \end{aligned} \tag{115}$$

Literature [15] report that the number of volume of pure hydrogen added is $\Delta V_{100\%}$=849.5 m^3 (30,000 cubic feet), while the amount by which the lifting power has been increased is 9986.25 N (2,245 pounds). Then for the volume of pure hydrogen added we have a percentage error equal to 0.963 %, whereas for the increment in the buoyancy, the percentage error is equal to 0.987 %.

Case study 8

A balloon in the hangar is to be filled to rise to a total altitude of 1524 m (5,000 feet) in bright sunshine. The observed temperature of the air is 294.2 K (70^{o}F). Assumeing that bright sunlight heats the gas to a temperature of 11.11 K (20^{o}F) above that of the surrounding air, we calculate the amount of hydrogen to be filled in the balloon in order that it will attain the desired altitude.

It is considered that the airship is in a "limp" state, then the weight of the gas W_{gas} remains constant. It is assumed that the total volume of the airship is V_T=6880.9 m^3 (243,000 cubic feet). At the ground the weight of the gas is given as

$$W_{gas} = \gamma_{gas_o} V_o, \tag{116}$$

where γ_{gas_o} and V_o (unknown value) are the gas specific weight and the initial volume of the airship at the ground (z=0 m), respectively. At the height z=1524 m, the weight of the gas is given as

$$W_{gas} = \gamma_{gas_z} V_z, \tag{117}$$

where γ_{gas_z} (unknown value) and V_z=V_T=6880.9 m^3 are the gas specific weight and the total volume of the airship at the height z=1524 m. Equating Eqs. (116) and (117), it can be obtained the value of the initial volume, that is the volume at the hangar V_o, hence we have

$$V_o = \left(\frac{\gamma_{gas_z}}{\gamma_{gas_o}}\right) V_T. \tag{118}$$

From the state equation γ_o is obtained as

$$\begin{aligned} \gamma_{gas_o} &= \frac{p_o}{B_{gas}T_{gas_o}} \\ &= \frac{101320}{(420.557)(294.2 + 11.11)} \\ &= 0.788 \text{ N/m}^3. \end{aligned} \tag{119}$$

The atmospheric pressure of the air (which the same as the gas) at the height z=1524 m, is calculated as, see Eq. (7)

$$
\begin{aligned}
p_z &= p_o \left[1 + \frac{dT_{air}}{dz}\frac{z}{T_{air_o}}\right]^{\frac{n}{n-1}} \\
&= 101320 \left[1 + \frac{(-0.0065)(1524)}{294.1}\right]^{\frac{1.2349}{(1.2349-1)}} \\
&= 84627.53 \text{ N/m}^2
\end{aligned}
\tag{120}
$$

The temperature of the air at the height z=1524 m, is calculated as, see Eq. (8)

$$
\begin{aligned}
T_{air_z} &= T_{air_o} + \frac{dT_{air}}{dz}z \\
&= 294.2 + (-0.0065)(1524) \\
&= 284.35 \text{ K}.
\end{aligned}
\tag{121}
$$

Then, the gas temperature at the height z=1524 m, is T_{gas_z}=T_{air_z}+11.11=295.463 K. With the values of p_z and T_{gas_z} it is possible to evaluate the gas specific weight at the height z=1524 m, that is

$$
\begin{aligned}
\gamma_{gas_z} &= \frac{p_z}{B_{gas}T_{gas_z}} \\
&= \frac{84627.53}{(420.557)(295.463)} \\
&= 0.681 \text{ N/m}^3.
\end{aligned}
\tag{122}
$$

Using this value in Eq. (118), the volume at the hangar can be evaluated as

$$
V_o = \left(\frac{0.681}{0.788}\right) 6880.9 = 5940.038 \text{ m}^3. \tag{123}
$$

Literature [15] report 5691.68 m^3 (201,000 cubic feet) as the volume of hydrogen required. Hence we have a percentage error equal to 4.36 %.

4 Conclusion

In this paper we presented several case studies that compare the results of our thermodynamic model with the "slide rule" results that were used by the US pilots of the airships at the beginning of twentieth century. The variations in the results went between less than 1% to slightly more than 4%. Which shows not only the accuaracy of our thermodynamic model but also confirms that the model included in the slide rule was quite reliable.

Along the history of the airships (since 1783), several tragic incidents have been present. It can be said that the airships born before time. However, with the new materials (such as titanium, kevlar, mylar, etc.), as well as the capabilities of the new computers to calculate the deformation of solid bodies and to successfully predict the behaviour of the atmospheric flow fields, it is hope that a new era of the giant airships is comming soon.

The hydrostatic theory shown in this paper is very simple and easy to understand, then it is doubtless that undergraduate students of physics and engineering, or the practicing engineers, after reading this paper can perform the calculation of the aerostatic behaviour of airships and outer space balloons. Some problems that were proposed to the North-American pilots of the "Blimps" by the designers of a slide rule have been successfully solved in the case studies.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

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